Math 249 Lecture 6 Notes

Daniel Raban

September 6, 2017

1 Basic Character Theory

1.1 Maschke's theorem and orthogonality

We start off with two fundamental theorems in the theory of characters. We will not prove them yet, but it is useful to first see their consequences.

Theorem 1.1 (Maschke). *Let G be finite and K be a field with characteristic 0. Every finite dimensional G-module V over K is a direct sum of irreducible G-modules (possibly with repeats).*

Theorem 1.2 (Orthogonality of characters). *Let K be an algebraically closed field or characteristic 0. The irreducible characters χ_{V_i} form an orthonormal basis of the space $X_G = \{\chi : G \to K : \chi$ constant on conjugacy classes$\}$, where the inner product is*

$$(\chi, \varphi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\varphi(g^{-1}).$$

Remark 1.1. If $K = \mathbb{C}$, then $\chi(g^{-1}) = \overline{\chi(g)}$. This is because g has finite order, so its matrix image under a representation must also have finite order; this matrix must have eigenvalues that multiply to 1 (i.e. they are roots of unity). The inverse of roots of unity is the same as their complex conjugate, so taking the trace of the matrix (the sum of the eigenvalues when we diagonalize) proves the claim. So the inner product is

$$(\chi, \varphi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\varphi(g)},$$

which is a Hermitian inner product.

Example 1.1. Let V be the usual representation of S_n on $\mathbb{C}^n$. $\chi_V(\sigma)$ equals the number of elements fixed by σ. Try $n = 4$. The conjugacy classes are

conjugacy class	1	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
number of elements	1	6	8	3	6
χ_V	4	2	1	0	0
$\chi_{\mathbb{1}}$	1	1	1	1	1

where $\chi_{\mathbb{1}}$ is the character of the trivial representation.

We can calculate $(\chi_V, \chi_V) = 2$, which means that $\chi_V = \chi_1 + \chi_2$, where χ_1, χ_2 are irreducible. Now calculate $(\chi_V, \chi_{\mathbb{1}}) = 1$, which means that $\chi_1 = \chi_{\mathbb{1}}$. Let $W = \{(x_1, \cdots, x_n) : \sum x_i = 0\}$. Then $(\chi_W, \chi_W) = 1$, meaning it is irreducible. We can calculate that $\chi_V = \chi_{\mathbb{1}} + \chi_W$.

Example 1.2. Let $G \circlearrowright \mathbb{C}G$. Then

$$\chi_{\mathbb{C}G}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e. \end{cases}$$

So for any G-module V, we can calculate

$$(\chi_V, \chi_{\mathbb{C}G}) = \frac{1}{|G|} \sum_g \chi_{\mathbb{C}G}(g^{-1}) \chi_V(g) = \frac{|G|}{|G|} \chi_V(e) = \dim V.$$

Then if $\chi_{\mathbb{C}G} \cong \bigoplus_i V_i^{\dim V_i}$ for irreducible V_i, then $|G| = \sum_i (\dim V_i)^2$.

1.2 Characters of tensor products, duals, and more

1.2.1 Tensor products

We want to introduce a "working definition" of the tensor product for our purposes.

Definition 1.1. Let K be a field, and let V and W be vector spaces. Then the *tensor product* $V \otimes_K W$ is the vector space with basis $\{v_i \otimes w_j\}$, where $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W, and such that $\cdot \otimes \cdot$ is bilinear.

For general vectors in V and W, we can form

$$(a_1 v_1 + \cdots + a_n v_n) \otimes (b_1 w_1 + \cdots + b_m w_m) = a_1 b_1 (v_1 \otimes w_1) + \cdots + a_n b_m (v_n \otimes w_m).$$

Given $\alpha : V_1 \to V_2$ and $\beta : W_1 \to W_2$, we can define $\alpha \otimes \beta : V_1 \otimes W_1 \to V_2 \otimes W_2$ as $v \times w \mapsto \alpha(v) \times \beta(w)$.

In terms of matrices, the matrix C obtained from the tensor product of the matrices A and B is $C_{(i,j),(i',j')} = A_{i,i'} B_{j,j'}$. We can express C in block matrix form as

$$C = \begin{bmatrix} A_{1,1}B & A_{1,2}B & \cdots \\ A_{2,1}B & A_{2,2}B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

What is on the diagonal of C? It is precisely all the products of the diagonal entries of elements from A with diagonal elements from B. So

$$\chi_{V\otimes W}(g) = \operatorname{tr}(g, V \otimes W) = \operatorname{tr}(g, V)\operatorname{tr}(g, W) = \chi_V(g)\chi_W(g).$$

1.2.2 Duals

Now consider the dual vector space V^* of a vector space G. Taking a matrix for V to the corresponding matrix for V^* gives us the transpose and reverses matrix multiplication. So when we make a representation of V^* from a representation of V, we replace g by g^{-1} and take the transpose. So

$$\chi_{V^*}(g) = \chi_V(g^{-1}).$$

1.2.3 Hom(V,W)

If V, W are G-modules, then $\operatorname{Hom}(V, W)$ is a G-module, where the action is $g \cdot \varphi = \rho_W(g)\varphi\rho_V(g)^{-1}$.

There is an isomorphism of G-modules $W \otimes V^* \cong \operatorname{Hom}(V, W)$ given by $w \otimes \lambda \mapsto \varphi_{w,\lambda}$, where $\varphi_{w,\lambda}(v) = \lambda(v) \cdot w$. This means that

$$\chi_{\operatorname{Hom}(V,W)}(g) = \chi_{W\otimes V^*}(g) = \chi_W(g)\chi_V(g^{-1}).$$

Now look at $(\chi_{\mathbb{1}}, \chi_{\operatorname{Hom}(V,W)})$. This is equal to $\dim \operatorname{Hom}(V, W)^G$, the dimension of the subspace of $\operatorname{Hom}(V, W)$ fixed by the action of G, because this is the number of times the trivial G-module shows up in the decomposition of $\operatorname{Hom}(V, W)$ into irreducible G-modules. This, in turn, is equal to $\dim \operatorname{Hom}_G(V, W)$, the dimension of the subspace of $\operatorname{Hom}(V, W)$ of maps φ such that $\rho_W(g)\varphi = \varphi\rho_V(g)$ for all $g \in G$ (equality follows from the definition of the action of G on $\operatorname{Hom}(V, W)$). We then get

$$\begin{aligned}
\dim \operatorname{Hom}_G(V, W) &= \dim \operatorname{Hom}(V, W)^G \\
&= (\chi_{\mathbb{1}}, \chi_{\operatorname{Hom}(V,W)}) \\
&= \frac{1}{|G|}\sum_g \chi_{\mathbb{1}}(g)\chi_{\operatorname{Hom}(V,W)}(g^{-1}) \\
&= \frac{1}{|G|}\sum_g \chi_W(g^{-1})\chi_V(g) \\
&= (\chi_V, \chi_W),
\end{aligned}$$

which gives us another interpretation of the inner product.